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The quantum mechanics of affine variables

José M. Isidro*

Dipartimento di Fisica "G. Galilei", Istituto Nazionale di Fisica Nucleare, Università di Padova, Via Marzolo 8, I-35131 Padova, Italy

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Abstract

We present a quantum-mechanical model for S-duality symmetries observed in the quantum theories of fields, strings and branes. Our formalism may be understood as the topological limit of Berezin's metric quantisation of the upper half-plane **H**, in that the metric dependence of Berezin's method has been removed. Being metric-free, our prescription makes no use of global quantum numbers. Quantum numbers arise only locally, after the choice of a local vacuum to expand around. Our approach may be regarded as a manifestly non-perturbative formulation of quantum mechanics, in that we take no classical phase space and no Poisson brackets as a starting point. The resulting quantum mechanics turns out to be that of the affine variables of quantum gravity. © 2002 Elsevier Science B.V. All rights reserved.

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1. Overview

1.1. Setup

The concept of *duality* plays a key role in recent important developments in the quantum theories of fields [1], string duality [2], M-theory and branes [3], M(atrix) theory [4], and the AdS/CFT correspondence [5]. Broadly speaking, under duality one understands a transformation of a given field or string theory, in a certain regime of the variables and parameters

* Tel.: +39-049-844-111; fax: +39-049-844-245.
E-mail address: isidro@pd.infn.it (J.M. Isidro).

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that define it, into a physically equivalent theory with different variables and parameters. The theories thus mapped into each other may be of apparently very different nature — e.g., the duality may exchange a field theory with a string theory. Alternatively, the duality may exchange the strong-coupling regime of a given theory with the perturbative regime of its dual theory, thus making the former more tractable. This latter form of mapping different theories goes under the name of S-duality. Often, what appears to be a highly non-trivial quantum excitation in a given field or string theory may well turn out to be a simple perturbative correction from the viewpoint of a theory dual to the original one. This suggests that what constitutes a quantum correction may be a matter of convention: the notion of *classical* versus *quantum* is relative to which theory the measurement is made from.

In view of these developments, Vafa [6] and other authors have suggested that quantum mechanics itself may need a revision if it is to accommodate, already from first principles, the notion of duality.

This state of affairs is reminiscent of general relativity. In fact a very interesting derivation of quantum mechanics from an equivalence principle has been given in [7,8]. In this formulation, conformal symmetry plays a key role.

Conformal quantum mechanics, as initiated in [9] and later supersymmetrised in [10], has also been the subject of renewed interest in connection with multi-black hole quantum mechanics (see [11,12] for extensive references).

1.2. Summary

Motivated by the above considerations, in this paper we develop a quantum mechanics that naturally incorporates a simple form of S-duality. The latter will be modelled on the conformal transformation of a complex variable $z \to -z^{-1}$. Due to the presence of conformal symmetry, our formalism may also be understood as the appropriate quantum mechanics for the affine variables of quantum gravity, where the affine algebra plays a significant role [13,14]. The generator of translations is represented by an operator with strictly positive spectrum. A similar feature will appear in our formalism in Section 3. When generalising the one-dimensional affine algebra to several dimensions [15], the generator of translations becomes a symmetric, positive-definite matrix degree of freedom. Such an object is well suited to describe the spatial part of the metric tensor. The coherent-state representation of the one-dimensional affine algebra has been studied in [16,17], and it has been generalised to several dimensions in [18].

The presence of conformal symmetry suggests considering a variable defined on Poincaré's upper half-plane $\mathbf{H}$. On the latter there exists the *holomorphic Fourier transformation* (HFT), which we intend to use as a technical tool for quantising an affine variable. The HFT relates a real co-ordinate on $\mathbf{R}$ to a complex momentum on $\mathbf{H}$. Alternatively, a real momentum can be HFT-transformed into a complex co-ordinate on $\mathbf{H}$. Position and momentum operators satisfying the Heisenberg algebra will be defined as dictated by the HFT. The Hilbert space of states will be identified explicitly. It will turn out to be larger than the standard $L^2(\mathbf{R})$ Hilbert space, as a consequence of the non-perturbative nature of our quantisation. We will explain how it eventually reduces to the usual $L^2(\mathbf{R})$. The wave function on $\mathbf{R}$ will be the restriction to the boundary of a holomorphic wave function whose natural domain will be $\mathbf{H}$. However, the quantum-mechanical operator Z corresponding to the classical variable $z \in \mathbf{H}$ will not be self-adjoint, so its physical interpretation requires some care. One can nonetheless make sense out of a non-self-adjoint operator Z. This is based on the fact that Z^2 admits a self-adjoint Friedrichs extension, whose square root is now self-adjoint.

Our formalism may be understood as a certain limit of Berezin's quantisation [19–22]. The latter relies on the metric properties of classical phase space $\mathcal{M}$, whenever $\mathcal{M}$ is a homogeneous Kähler manifold [23,24]. In Berezin's method, quantum numbers arise naturally from the metric on $\mathcal{M}$. The semiclassical regime is then identified with the regime of large quantum numbers. Our method may be regarded as the topological limit of Berezin's quantisation, in that the metric dependence has been removed. Topological gravity has in fact a long history [25]. As a consequence of this topological nature our quantisation exhibits some added features. Quantum numbers are not originally present in our prescription; they appear only after a vacuum has been chosen, and even then they are local in nature, instead of global. Hence our procedure may be thought of as a manifestly non-perturbative formulation of quantum mechanics, in that we take no classical phase space and no Poisson brackets as our starting point, i.e., we do not deform a classical theory into its quantum counterpart, as in deformation quantisation [26–34].

On the upper half-plane $\mathbf{H}$ there is an isometric action of the group $SL(2, \mathbf{R})$. Berezin's metric method, applied to $\mathbf{H}$, yields a Hilbert space of states $\mathcal{H}$ that provides a representation space for $SL(2, \mathbf{R})$. However, our approach makes no use of the metric properties of $\mathbf{H}$. Correspondingly, we have no representation of $SL(2, \mathbf{R})$ as a Hilbert space of states. Also in this sense our quantisation is topological, as opposed to Berezin's metric approach.

1.3. Outline

This article is organised as follows. The HFT is presented in technical detail in Section 2. Section 3 develops a quantum mechanics based on the HFT. Special emphasis is placed on a technical analysis of the spectral properties of operators. Section 4 is devoted to a physical interpretation of our formalism. We discuss why the non-isospectrality of the HFT allows for non-trivial dualities that are necessarily absent in the context of Schrödinger pairs, as in standard quantum mechanics. We also explain the physical meaning of the non-self-adjoint operator Z, the choice of a vacuum and the breaking of $SL(2, \mathbf{R})$ to the affine group of quantum gravity, as well as the topological character of this quantum mechanics. Using an $SL(2, \mathbf{R})$ action on the operators, we exhibit how to implement an S-duality between strong quantum effects and semiclassical corrections in our framework. Finally, in Section 5 we make some closing comments.

2. The holomorphic Fourier transformation (HFT)

By analogy with Berezin's quantisation [19–24], we need a space of analytic functions on the upper half-plane $\mathbf{H}$ as our Hilbert space of states $\mathcal{H}$. A key observation is that the holomorphic Fourier transformation, summarised below, provides such a space in a natural way [35–37].

Let $F_\psi \in L^2(0,\infty)$. For $z = x + \mathrm{i}y \in \mathbf{H}$, the function ψ is defined as

$$\psi(z) = \frac{1}{\sqrt{2\pi}}\int_0^\infty \mathrm{d}t\, F_\psi(t)\,\mathrm{e}^{\mathrm{i}tz}, \tag{1}$$

the integral understood in the sense of Lebesgue, is holomorphic on $\mathbf{H}$. Its restrictions to horizontal straight lines $y = \text{const.} > 0$ in $\mathbf{H}$ are a bounded set in $L^2(\mathbf{R})$.

Conversely, let ψ be holomorphic on $\mathbf{H}$, and assume that

$$\sup_{0<y<\infty}\int_{-\infty}^\infty \mathrm{d}x|\psi(x+\mathrm{i}y)|^2 = C < \infty. \tag{2}$$

Then the function F_ψ is defined by

$$F_\psi(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \mathrm{d}z\, \psi(z)\,\mathrm{e}^{-\mathrm{i}tz}, \tag{3}$$

the integration being along any horizontal straight line $y = \text{const.} > 0$ in $\mathbf{H}$ satisfies the following basic properties. $F_\psi(t)$ is independent of the particular horizontal line $y = \text{const.} > 0$ chosen. Moreover, $F_\psi \in L^2(0,\infty)$, and for any $z \in \mathbf{H}$, Eq. (1) holds, with

$$\int_0^\infty \mathrm{d}t|F_\psi(t)|^2 = C. \tag{4}$$

We call F_ψ the holomorphic Fourier transform of ψ.

Some features of the HFT on $\mathbf{H}$ are worth mentioning. Let $\Omega(\mathbf{H})$ denote the space of all holomorphic functions on $\mathbf{H}$, and let $\Omega_0(\mathbf{H})$ denote the proper subspace of all $\psi \in \Omega(\mathbf{H})$ such that the supremum C introduced in (2) is finite. Then C defines a squared norm $\|\psi\|^2$ on $\Omega_0(\mathbf{H})$. The subspace $\Omega_0(\mathbf{H})$ is complete with respect to this norm. This norm is Hilbert, i.e., it verifies the parallelogram identity. Hence the scalar product $\langle\varphi|\psi\rangle$ defined on $\Omega_0(\mathbf{H})$ through

$$4\langle\varphi|\psi\rangle = \|\psi+\varphi\|^2 - \|\psi-\varphi\|^2 + \mathrm{i}\|\psi+\mathrm{i}\varphi\|^2 - \mathrm{i}\|\psi-\mathrm{i}\varphi\|^2 \tag{5}$$

turns the complete normed space $\Omega_0(\mathbf{H})$ into a Hilbert space with respect to the scalar product (5). In fact, via the HFT, the subspace $\Omega_0(\mathbf{H})$ is isometrically isomorphic to the Hilbert space $L^2(0,\infty)$.

3. Quantum mechanics from the HFT

3.1. The space of quantum states

Section 2 allows us to identify the Hilbert space of states $\mathcal{H}$ of our quantum mechanics. In the representation in which the wave function is $F_\psi(t)$ we have $\mathcal{H} = L^2(0,\infty)$, while in its HFT-transformed representation $\psi(z)$ we have $\mathcal{H} = \Omega_0(\mathbf{H})$. After the choice of a vacuum in Section 4.2 and the introduction of the boundary wave function in Section 4.3, we will see the emergence of the usual Hilbert space $L^2(\mathbf{R})$.

For definiteness, we choose the complex variable $z \in \mathbf{H}$ to stand for the momentum p, with the real variable $t \in (0, \infty)$ standing for the co-ordinate q. Then the HFT reads

$$\psi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_0^\infty \mathrm{d}q\, F_\psi(q)\, \mathrm{e}^{(\mathrm{i}/\hbar)qp}, \quad F_\psi(q) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^\infty \mathrm{d}p\, \psi(p)\, \mathrm{e}^{-(\mathrm{i}/\hbar)qp}. \tag{6}$$

3.2. Position and momentum

In co-ordinate representation, we define position and momentum operators Q and P:

$$(QF_\psi)(q) = qF_\psi(q), \qquad (PF_\psi)(q) = \mathrm{i}\hbar \frac{\mathrm{d}F_\psi}{\mathrm{d}q}. \tag{7}$$

Eq. (6) implies that their momentum representation is

$$(Q\psi)(p) = -\mathrm{i}\hbar \frac{\mathrm{d}\psi}{\mathrm{d}p}, \qquad (P\psi)(p) = p\,\psi(p). \tag{8}$$

Irrespective of the representation chosen we have the Heisenberg algebra

$$[P, Q] = \mathrm{i}\hbar\,\mathbf{1}. \tag{9}$$

On the domain

$$D(Q) = \left\{ F_\psi \in L^2(0, \infty) : \int_0^\infty \mathrm{d}q\, q^2 |F_\psi(q)|^2 < \infty \right\}, \tag{10}$$

which is dense in $\mathcal{H}$, the operator Q is symmetric,

$$\langle F_\psi | Q | F_\varphi \rangle^* = \langle F_\varphi | Q | F_\psi \rangle. \tag{11}$$

A closed, symmetric, densely defined operator admits a self-adjoint extension if and only if its defect indices $d_\pm$ are equal. Moreover, such an operator is essentially self-adjoint if and only if its defect indices are both zero [35–37]. The operator Q turns out to be essentially self-adjoint, with point, residual and continuous spectra given by

$$\sigma_\mathrm{p}(Q) = \emptyset, \qquad \sigma_\mathrm{r}(Q) = \emptyset, \qquad \sigma_\mathrm{c}(Q) = [0, \infty). \tag{12}$$

The properties of the momentum operator P are subtler. One finds

$$\langle F_\psi | P | F_\varphi \rangle^* = \mathrm{i}\hbar F_\psi(0) F_\varphi^*(0) + \langle F_\varphi | P | F_\psi \rangle, \tag{13}$$

so P is symmetric on the domain

$$D(P) = \left\{ F_\psi \in L^2(0, \infty) : F_\psi \text{ abs. cont.}, \int_0^\infty \mathrm{d}q \left| \frac{\mathrm{d}F_\psi}{\mathrm{d}q} \right|^2 < \infty,\ F_\psi(0) = 0 \right\} \tag{14}$$

(F_ψ is absolutely continuous). The adjoint $P^\dagger$ also acts as $\mathrm{i}\hbar(\mathrm{d}/\mathrm{d}q)$, with a domain $D(P^\dagger)$:

$$D(P^\dagger) = \left\{ F_\psi \in L^2(0, \infty) : F_\psi \text{ abs. cont.}, \int_0^\infty \mathrm{d}q \left| \frac{\mathrm{d}F_\psi}{\mathrm{d}q} \right|^2 < \infty \right\}, \tag{15}$$

where the boundary condition $F_\psi(0) = 0$ has been lifted. On the space $L^2(0, \infty)$ we have $d_+(P) = 0$, $d_-(P) = 1$. We conclude that P admits no self-adjoint extension. Its point, residual and continuous spectra are

$$\sigma_{\mathrm{p}}(P) = \emptyset, \quad \sigma_{\mathrm{r}}(P) = \mathbf{H} \cup \mathbf{R}, \quad \sigma_{\mathrm{c}}(P) = \emptyset. \tag{16}$$

The domain $D(P)$ is strictly contained in $D(P^\dagger)$. This implies that the operators $P_x = \frac{1}{2}(P + P^\dagger)$ and $P_y = (P - P^\dagger)/2\mathrm{i}$ which one would naively construct out of P are ill defined. There is no way to define self-adjoint operators P_x and P_y corresponding to the classical momenta p_x and p_y. This is compatible with the fact that the defect indices of P being unequal, P does not commute with any complex conjugation on $\mathcal{H}$ [35–37]. However, we will see presently that one can make perfectly good sense of a quantum mechanics whose momentum operator P admits no self-adjoint extension. We defer issues like measurements of P and Heisenberg's uncertainty principle until Section 4.2. Quadratic terms in P are technically simpler, and will be dealt with first.

With our choice of domain $D(P)$, which makes P symmetric, P^2 is also symmetric. One proves that $d_-(P^2) = 1 = d_+(P^2)$. Hence P^2, although not essentially self-adjoint, admits a self-adjoint extension. A popular choice is the Friedrichs extension [35–37]. Given an operator A, this extension is characterised by a boundedness condition

$$\langle\varphi|A|\varphi\rangle \geq -\alpha\|\varphi\|^2 \quad \forall\varphi \in D(A) \tag{17}$$

for a certain $\alpha \geq 0$. Now the operator P^2 admits a Friedrichs extension P_{F}^2 with a lower bound $\alpha = 0$:

$$\langle F_\varphi|P_{\mathrm{F}}^2|F_\varphi\rangle \geq 0 \quad \forall F_\varphi \in D(P_{\mathrm{F}}^2). \tag{18}$$

The point, residual and continuous spectra of this extension are

$$\sigma_{\mathrm{p}}(P_{\mathrm{F}}^2) = \emptyset, \qquad \sigma_{\mathrm{r}}(P_{\mathrm{F}}^2) = \emptyset, \qquad \sigma_{\mathrm{c}}(P_{\mathrm{F}}^2) = [0, \infty). \tag{19}$$

Now the crucial point is that the square root of the Friedrichs extension allows us to define a self-adjoint momentum operator. Let us define the new operator $P_{\surd}$

$$P_{\surd} = \sqrt{P_{\mathrm{F}}^2}. \tag{20}$$

$P_{\surd}$ is self-adjoint, with a domain $D(P_{\surd})$ uniquely determined by the spectral decomposition of P [35–37]. The point, residual and continuos spectra of $P_{\surd}$ are

$$\sigma_{\mathrm{p}}(P_{\surd}) = \emptyset, \quad \sigma_{\mathrm{r}}(P_{\surd}) = \emptyset, \quad \sigma_{\mathrm{c}}(P_{\surd}) = [0, \infty). \tag{21}$$

We observe that taking the Friedrichs extension does not commute with the square root. The operator $P_{\surd}$ enjoys the properties of being linear in p and having the right commutator (9) with the position operator.

3.3. $SL(2, \mathbf{R})$-*Transformation of the operators*

We can reparametrise the co-ordinate $z \in \mathbf{H}$ by means of a Möbius transformation $z \to \tilde{z} = (az + b)(cz + d)^{-1}$, with $ad - bc = 1$. We now consider the HFT written as

$$\tilde{\psi}(\tilde{p}) = \frac{1}{\sqrt{2\pi\hbar}}\int_0^\infty \mathrm{d}\tilde{q}\, \tilde{F}_{\tilde{\psi}}(\tilde{q})\, \mathrm{e}^{(\mathrm{i}/\hbar)\tilde{q}\tilde{p}} \quad \tilde{F}_{\tilde{\psi}}(\tilde{q}) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^\infty \mathrm{d}\tilde{p}\, \tilde{\psi}(\tilde{p})\, \mathrm{e}^{-(\mathrm{i}/\hbar)\tilde{q}\tilde{p}}, \tag{22}$$

where $\tilde{q} \in (0, \infty)$ is the variable dual to $\tilde{p}$ under (22). One can define co-ordinate and momentum operators $\tilde{Q}$ and $\tilde{P}$ satisfying the Heisenberg algebra (9). Hence this is a canonical transformation from (q, p) to $(\tilde{q}, \tilde{p})$. The transformed operators $\tilde{Q}$ and $\tilde{P}$ have the same spectra as before.

4. Discussion

4.1. Non-isospectrality of the HFT

The standard Fourier transformation maps (a subspace of) $L^2(\mathbf{R})$ into (a subspace of) $L^2(\mathbf{R})$. It is also an isospectral transformation between self-adjoint operators. In the context of the standard Fourier transformation on $L^2(\mathbf{R})$, co-ordinate and momentum are sometimes referred to as a *Schrödinger pair*.

On the contrary, the HFT is not an isospectral transformation; Q and P do not have identical spectra. Furthermore, the very choice of the dynamical variable to be represented by complex variable z of the HFT is a non-trivial choice in itself. These properties allow for non-trivial dualities that are necessarily absent in the context of Schrödinger pairs.

4.2. The choice of a local vacuum

The difficulties due to the fact that one of the two canonical operators (Q, P) admits no self-adjoint extension can be overcome by *the choice of a vacuum* to expand around. Under the latter we understand the choice of either z or $\tilde{z} = -z^{-1}$ as the classical co-ordinate on $\mathbf{H}$ to be quantised into the operator Z or $\tilde{Z} = -Z^{-1}$. After the choice of a vacuum, the $SL(2, \mathbf{R})$ symmetry is reduced to translations and dilatations, leaving the affine group only.

For definiteness, let us choose the vacuum corresponding to the classical variable z, in the picture in which the quantum operator Z is the momentum P. Then the construction of Section 3 leads to a pair of self-adjoint operators $(Q, P_{\surd})$. They are almost canonically conjugated in the sense that while satisfying the Heisenberg algebra (9), the exchange between co-ordinate and momentum is not performed directly at the level of $(Q, P_{\surd})$ by means of the usual Fourier transformation. Rather, $(Q, P_{\surd})$ have to be lifted back to their HFT ancestors (Q, P) in order to exchange them. Apart from this technicality, the operators $(Q, P_{\surd})$ meet the usual quantum-mechanical requirements concerning the measurement process and the Heisenberg uncertainty principle. The vacuum $|0_z\rangle$, and the corresponding local quantum numbers n_z obtained upon expansion around it, will certainly differ from the vacuum $|0_{\tilde{z}}\rangle$ and the quantum numbers $n_{\tilde{z}}$ obtained from the classical variable $\tilde{z} = -z^{-1}$. So this choice of a vacuum is *local in nature*, in that it is linked to a specific choice of co-ordinate. The coherent states constructed around $|0_z\rangle$ are not coherent from the viewpoint of $|0_{\tilde{z}}\rangle$. We conclude that this quantum mechanics does not allow for globally defined coherent states such as those of Berezin's quantisation.

This choice of a vacuum is reminiscent of M-theory and the (perturbatively) different string theories it unifies [2,3]. The eleventh dimension of M-theory, as opposed to the 10 critical dimensions of the type IIA string appears in the passage to the strong-coupling limit. In doing so, one succeeds in incorporating the known dualities between different perturbative strings. In our context, the HFT canonically relates the two real dimensions of the upper half-plane **H** to the one real dimension of the real axis **R**. The extra dimension present in the HFT disappears once a vacuum has been chosen through the self-adjoint operator $P_{\surd}$.

4.3. *The wave function on the boundary*

After a vacuum has been chosen, the connection with standard quantum mechanics can be made more explicit by exhibiting the usual Hilbert space $L^2(\mathbf{R})$ emerge from our approach as follows. Let us consider the picture (dual to that of Section 3) in which the complex variable z is the co-ordinate q. For a holomorphic wave function $\psi(q) = \psi(x + \mathrm{i}y)$ satisfying condition (2), a boundary wave function $\psi_{\mathrm{b}}(x) \in L^2(\mathbf{R})$ exists such that [35–37]

$$\lim_{y\to 0}\int_{-\infty}^{\infty}\mathrm{d}x\,|\psi(x+\mathrm{i}y)-\psi_{\mathrm{b}}(x)|^2 = 0. \tag{23}$$

So while the requirement of L^2-integrability of standard quantum mechanics is maintained, the HFT extends the wave function $\psi_{\mathrm{b}}(x) \in L^2(\mathbf{R})$ of a particle on the boundary of **H** to a holomorphic $\psi(q)$ defined on the entire upper half-plane.

4.4. *A topological quantum mechanics*

Berezin's quantisation relied heavily on the metric properties of classical phase space. The semiclassical limit could be defined as the regime of large quantum numbers. The very existence of quantum numbers was a consequence of the metric structure.

On the contrary, the quantum mechanics developed here is completely independent of the metric properties of the upper half-plane **H**. Quantisation in terms of the HFT is *topological*, in that it does not know about the Poincaré metric $\mathrm{d}s^2 = (\mathrm{d}x^2+\mathrm{d}y^2)/y^2$. Indeed, the absence of a metric prevents us from writing an integration measure as in Berezin's quantisation. The supremum in Eq. (2) reflects this fact. Along any horizontal line $y = \text{const.} > 0$ one effectively observes a constant Euclidean metric; in order to detect the negative curvature of the Poincaré metric one needs to displace along y. The HFT correctly captures this property. Furthermore, a "thickening" of the real line **R** to the upper half-plane **H** should not feel the presence of the Poincaré metric on **H**, if it is to describe quantum mechanics on **R**. This is compatible with the interpretation of the wave function given in Section 4.3.

We therefore have a quantum mechanics that is free of *global* quantum numbers. The latter appear only after the choice of a *local* vacuum. The logic could be summarised as follows:

1. The fact that this quantum mechanics is topological implies the absence of a metric.
2. The absence of a metric implies the absence of global quantum numbers.

3. The absence of global quantum numbers implies the impossibility of globally defining a semiclassical regime. The latter exists only locally.

Actually, as our starting point we have no classical phase space at all, and no Poisson brackets to quantise into commutators. This may be regarded as a manifestly non-perturbative formulation of quantum mechanics, as required in [6]. The sections that follow elaborate on this point further.

4.5. Classical vs. quantum

Next we state a proposal to accommodate a simple form of S-duality into our framework. To be concrete, we assume that the required duality is $SL(2, \mathbf{R})$. In fact this group (or subgroups thereof) is ubiquitous in field and string duality. From the HFT we have developed a quantum mechanics that is conceptually as close as possible to the standard one, while at the same time incorporating the desired duality. By this we do not mean having a representation of $SL(2, \mathbf{R})$ as the Hilbert space of states. In fact, Berezin's quantisation does precisely that [19–22]. Rather, we have implemented a particularly relevant $SL(2, \mathbf{R})$ transformation, the inversion $z \to \tilde{z} = -z^{-1}$, on the quantum operator Z corresponding to the classical variable z. If Z is taken to represent the momentum P, the effect is that of transforming Planck's constant as $\hbar \to -\hbar^{-1}$. This can be interpreted as an exchange of the semiclassical with the strong quantum regime. In this context, $\hbar$ is best thought of as a dimensionless deformation parameter, as in deformation quantisation [26–34]. This duality symmetry is not implemented in ordinary quantum mechanics.

The quantum mechanics based on the HFT naturally incorporates this duality under a single theory. Different limits of the latter yield different regimes. Let us start from the classical variable $z \in \mathbf{H}$ and choose the corresponding non-self-adjoint quantum operator Z to be the momentum P. We can compute quantum effects to $O(\hbar)$, which one would call semiclassical *from the viewpoint of the quantum theory corresponding to the classical variable* z. Strong quantum effects, that will be of $O(-\hbar^{-1})$ *from the viewpoint of the original theory*, will appear to be simple semiclassical corrections of $O(\hbar)$ *from the viewpoint of the dual quantum theory corresponding to the classical variable* $\tilde{z} = -z^{-1}$.

5. Concluding remarks

In this article, we have tried an approach to quantum mechanics that is not primarily based on the quantisation of a given classical dynamics. In such an approach one does not take a classical theory as a starting point. Rather, quantum mechanics itself is more fundamental, in that its classical limit or limits (possibly more than one) follow from a parent quantum theory. One may regard such an approach as a formulation of quantum mechanics in the sense claimed by Vafa [6]: quantum corrections may depend on the observer, and semiclassical expansions do not have an absolute, i.e., co-ordinate-free meaning.

A key point in our presentation is the interplay between quantum mechanics and geometry. This bears out the notion that geometry is dynamical, i.e., it possesses physical degrees of freedom. As such this idea is of course not new. The novelty of our approach is its manifestly non-perturbative character. As has been pointed out in [38–40], a non-perturbative approach carries no background metric at all. Happily, this turns out to be the case in our approach as well: rather than a metric-based quantum mechanics we have a metric-free quantum mechanics. Metric-free theories usually go by the name of topological theories. The latter have been studied in an approach to quantum gravity from a diffeomorphism-invariant viewpoint [41]. This raises the exciting possibility that quantising gravity (outside the realm of string theory) and rendering the notion of duality compatible with quantum mechanics may be one and the same thing!

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